ON THE THEORY OF STABILITY OF A PLASMA IN A STRONG LONGITUDINAL MAGNETIC FIELD, VARIABLE ALONG THE AXIS OF SYMMETRY

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The investigation of stability of plasma string in a longitudinal magnetic field which is periodically changing in the direction of the axis of symmetry, is of interest because a single element of periodicity of such a system represents a trap with magnetic constrictions. The most dangerous perturbations are those which weakly distort the magnetic field, the so called fluted instabilities [1 to 3]. Using the method of kinetic equations, Rosenbluth and others in [4] examined the effect of stabilization of fluted instabilities which arise for sufficiently large magnitudes of the Larmor radius of ions. The problem of plasma stability in a longitudinal magnetic field alternating along the axis of the system was solved in [4] in an approximate manner by replacing the destabilizing effect which is related to the curvature of magnetic lines of force (arching away from the plasma) by a corresponding gravitational effect. Roberts and Taylor in [5] and Rudakov in [6] demonstrated that the stabilizing effect of Rosenbluth and others in [4] can be derived from a system of magnetohydrodynamic equations if terms which characterize the influence of the finite magnitude of Larmor radius of ions are taken into account in the viscous tension tensor. This "magnetic viscosity" remains in the absence of collisions, it does not lead to a dissipation of energy.

In the present work the stability problem is examined by the method of normal oscillations on the basis magnetohydrodynamic equations taking into account "magnetic viscosity". It is assumed that the plasma pressure is small compared to the magnetic pressure and the only perturbations examined are those which do not distort the magnetic field (in the first approximation).

In sections 1 to 4 the stability is investigated for infinitely small Larmor radius where the plasma behavior can be described by mean of a system of equations for a one-fluid conducting medium. In sections 1 and 2 it is shown that the problem reduces to solving a common second order differential equation in which the coefficients are average values taken along a magnetic line of force. For a complete determination of these coefficients it is necessary to find the solution of one more differential equation (for a given distribution of the magnetic field in the steady state). In section 3 the stability of a plasma string is examined the diameter of which is much smaller than the length of one element of periodicity. In section 4 the stability is investigated with respect to fluted instabilities which slowly grow with time. Results obtained in sections 3 and 4 agree with conclusions in [1 to 3].

Section 5 is devoted to an investigation of plasma stability with uniform cross-sectional temperatures of ions and electrons taking into account effects due to large magnitude of Larmor radius of ions. It is assumed that the curvature of lines of the magnetic force is a minor parameter. The solutions for the problem are expressed in well known functions for the distribution of density and pressure which decay exponentially with distance from the axis of symmetry. It is shown that stabilization of fluted instabilities is possible (with the exception of some perturbations $m = 1$). Results of calculations are in agreement with approximate computations in [4].

1. Basic equations for the case of small Larmor radius of ions. The system of magnetohydrodynamic equations which describe the behavior of an ideally conducting one-fluid medium in a magnetic field $\mathbf{H}$ has the form

$$\rho \frac{d\mathbf{v}}{dt} = -\nabla p - \frac{1}{4\pi}\mathbf{H} \times \text{rot}\,\mathbf{H}, \qquad \frac{d}{dt} = \frac{\partial}{\partial t} + \mathbf{v}\cdot\nabla \tag{1.1}$$

$$\frac{\partial \mathbf{H}}{\partial t} = \text{rot}\,(\mathbf{v}\times\mathbf{H}), \qquad \text{div}\,\mathbf{H} = 0 \tag{1.2}$$

$$\frac{\partial \rho}{\partial t} + \text{div}\,\rho\mathbf{v} = 0 \tag{1.3}$$

$$\frac{d}{dt}(p\rho^{-\gamma}) = 0, \qquad \gamma = \text{const} \tag{1.4}$$

Here $\mathbf{v}$ is the velocity, p the pressure, and ρ the density of the medium. For the equilibrium case we let $\mathbf{v} = 0$, $\mathbf{H} = \mathbf{H}(\mathbf{r})$, then from (1.1) and (1.2) we obtain

$$\nabla p = -\frac{1}{4\pi}\mathbf{H} \times \text{rot}\,\mathbf{H}, \qquad \text{div}\,\mathbf{H} = 0 \tag{1.5}$$

We will now examine the stability of this state with respect to perturbations which have a time dependence of the form $\exp(i\omega t)$. The quantities obtained in the perturbed state will be

$$p + p^*,\ \rho + \rho^*,\ \mathbf{H} + \mathbf{H}^*,\ \mathbf{v}^* = i\omega\boldsymbol{\xi},$$

where the asterisk indicates perturbations. The linearized system (1.1) to (1.4) has the form

$$\mathbf{H} \times \text{rot}\,\mathbf{H}^* = \mathbf{K} \tag{1.6}$$

$$\mathbf{H}^* = \text{rot}\,(\boldsymbol{\xi} \times \mathbf{H}), \qquad \text{div}\,\mathbf{H}^* = 0 \tag{1.7}$$

$$\rho^* = -\rho\,\text{div}\,\boldsymbol{\xi} - \boldsymbol{\xi}\cdot\nabla\rho, \qquad p^* = -\gamma p\,\text{div}\,\boldsymbol{\xi} - \boldsymbol{\xi}\cdot\nabla p \tag{1.8}$$

$$\mathbf{K} = 4\pi\rho\omega^2\boldsymbol{\xi} - 4\pi\nabla p^* - \mathbf{H}^* \times \text{rot}\,\mathbf{H} \tag{1.9}$$

For the condition $8\pi p \ll H^2$ Equation (1.6) can be reduced to a simple form [2]. If forceless configurations are not examined then it follows from (1.5) that $|\text{rot}\,\mathbf{H}| \sim 4\pi p / r_0 H$, where r_0 is the characteristic dimension. In Equation (1.6) the member on the left hand side is H^2/p times greater than $|\mathbf{K}|$, therefore it is necessary to assume as a first approximation

$$\mathbf{H} \times \text{rot}\,\mathbf{H}^* = 0 \tag{1.10}$$

Now, in the left hand side of (1.6) there is an undetermined quantity and it must be eliminated. In other words, here terms of the subsequent order with respect to the small parameter p/H^2 are essential and it is necessary to utilize the condition of solvability of (1.6) [2]. From scalar multiplication of (1.6) by $\mathbf{H}$ we obtain

$$\mathbf{H}\cdot\mathbf{K} = 0 \tag{1.11}$$

Further we find

$$\text{rot } \mathbf{H}^* = \mathbf{G} + \alpha\mathbf{H}, \qquad \mathbf{G} = \frac{1}{H^2}\mathbf{K}\times\mathbf{H} \tag{1.12}$$

$\alpha(\mathbf{r})$ is an arbitrary single-valued function of the coordinates. By applying the operation div to first Equation in (1.12), we obtain

$$\text{div}\left(\frac{\mathbf{H}\times\mathbf{K}}{H^2}\right) = \mathbf{H}\cdot\nabla\alpha \tag{1.13}$$

Let s be the arc length of some force line in the unperturbed magnetic field **H**. Multiplying (1.13) by ds/H and intergrating with respect to s, we obtain on the right hand side the difference in the values of α over the limits of integration. We will examine configurations with closed magnetic lines of force or configurations which have periodically repetitive form and perturbations with the same period. Then along the entire line of force or along one element of periodicity the integral

$$\oint \frac{1}{H}\text{div}\left(\frac{\mathbf{H}\times\mathbf{K}}{H^2}\right)ds = 0 \tag{1.14}$$

By a somewhat different method this Equation was derived by Kadomtsev in [2] for the case where the perturbation of the magnetic field is small. The system of equations for the stability problem, as follows from (1.7) to (1.11) and (1.14), has the form

$$\mathbf{H}^* = \text{rot}\,(\boldsymbol{\xi}\times\mathbf{H}), \qquad \mathbf{H}\times\text{rot } \mathbf{H}^* = 0 \tag{1.15}$$

$$\omega^2\rho\mathbf{H}\cdot\boldsymbol{\xi} + \mathbf{H}\cdot\nabla(\gamma p\,\text{div}\,\boldsymbol{\xi} + \boldsymbol{\xi}\cdot\nabla p) - \mathbf{H}^*\cdot\nabla p = 0 \tag{1.16}$$

$$\oint\frac{1}{H}\text{div}\left\{\frac{\mathbf{H}}{H^2}\times[\omega^2\rho\boldsymbol{\xi} + \nabla(\gamma p\,\text{div}\,\boldsymbol{\xi} + \boldsymbol{\xi}\cdot\nabla p) - \frac{1}{4\pi}\mathbf{H}^*\times\text{rot }\mathbf{H}]\right\}ds = 0 \tag{1.17}$$

2. Plasma in an extended magnetic field which is varying along the axis of symmetry. In cylindrical coordinates r, φ and z let the equilibrium magnetic field be symmetrical with respect to the axis and let the field not have a φ component. It is possible to introduce such a function ψ that

$$\mathbf{H} = \text{rot}\left(\mathbf{i}_\varphi\frac{\psi}{r}\right) = -\frac{1}{r}\mathbf{i}_\varphi\times\nabla\psi, \qquad |\mathbf{i}_\varphi| = 1, \qquad (\psi)_{r=0} = 0 \tag{2.1}$$

Since $\mathbf{H}\cdot\nabla\psi = 0$, the lines of force lie on surfaces with ψ =const, where $2\pi\psi$ is equal to the flow within the surface ψ =const. We introduce curvilinear orthogonal coordinates $x_1 = \psi$, $x_2 = \varphi$, $x_3 = \chi$; an element of volume is equal to $J\,d\psi\,d\varphi\,d\chi$, for Lamé coefficients we obtain [3]

$$h_1 = \frac{1}{rH}, \qquad h_2 = r, \qquad h_3 = JH \tag{2.2}$$

In the transformation from cylindrical to curvilinear coordinates following Formulas can be used

$$\frac{\partial r}{\partial\psi}\frac{\partial r}{\partial\chi} + \frac{\partial z}{\partial\psi}\frac{\partial z}{\partial\chi} = 0$$

$$J = r\left\{\left[\left(\frac{\partial r}{\partial\psi}\right)^2 + \left(\frac{\partial z}{\partial\psi}\right)^2\right]\left[\left(\frac{\partial r}{\partial\chi}\right)^2 + \left(\frac{\partial z}{\partial\chi}\right)^2\right]\right\}^{1/2} = \frac{r}{|\nabla\psi|\cdot|\nabla\chi|} \tag{2.3}$$

$$\frac{1}{rH} = \left\{\left(\frac{\partial r}{\partial \psi}\right)^2 + \left(\frac{\partial z}{\partial \psi}\right)^2\right\}^{1/2} = \frac{1}{|\nabla \psi|} \tag{2.3}$$

For the subsequent presentation we assume that the field is periodic along with the period $2l$. Within one element of periodicity z and also χ vary from $-l$ to l. Equilibrium Equation (1.5) leads to relationship [3]

$$\frac{\partial p}{\partial \chi} = 0, \qquad \frac{\partial p}{\partial \psi} = -\frac{1}{4\pi J}\frac{\partial J H^2}{\partial \psi} \tag{2.4}$$

We will also assume that density ρ is a function of one variable ψ.

Now we will examine the problem of stability assuming that plasma behavior can be described by a system of equations of one-fluid approximation (1.1) to (1.4). Introducing the notation

$$X = rH\xi_\psi, \qquad Y = \frac{im}{r}\xi_\varphi, \qquad Z = \frac{1}{H}\xi_\chi \tag{2.5}$$

we obtain from Equations (1.15)

$$H_\psi{}^* = \frac{1}{rJH}\frac{\partial X}{\partial \chi}, \qquad H_\varphi{}^* = \frac{r}{imJ}\frac{\partial Y}{\partial \chi}, \qquad H_\chi{}^* = -H\left(\frac{\partial X}{\partial \psi} + Y\right) \tag{2.6}$$

$$\frac{\partial X}{\partial \psi} + Y - \frac{1}{m^2 JH^2}\frac{\partial}{\partial \chi}\left(\frac{r^2}{J}\frac{\partial Y}{\partial \chi}\right) = 0, \quad \frac{\partial}{\partial \chi}\left[JH^2\left(\frac{\partial X}{\partial \varphi} + Y\right)\right] + \frac{\partial}{\partial \chi}\left(\frac{1}{r^2 JH^2}\frac{\partial X}{\partial \chi}\right) = 0$$

It is assumed here that $\xi = \xi_{(1)}(\psi, \chi)\, e^{im\varphi}$. In the following text we omit the factor $e^{im\varphi}$

We will limit ourselves to a study of instabilities for which the perturbation of the magnetic field $\mathbf{H}^* = 0$.

$$X = X(\psi), \qquad Y = -\frac{dX}{d\psi} \tag{2.7}$$

Equations (2.6) are satisfied. It follows from the equality (1.16)

$$\gamma p \frac{\partial}{\partial \chi}\left(\frac{1}{J}\frac{\partial S}{\partial \chi}\right) + \omega^2 \rho J H^2 S = -\gamma p \frac{\partial^2 \ln J}{\partial \chi \partial \psi} \qquad (Z = S(\psi, \chi)\, X) \tag{2.8}$$

We will examine perturbations which are periodic with respect to χ with a period $2l$. The function S will be periodic with the same period. This circumstance permits the determination of arbitrary constants in the integration of (2.8).

After substitution of Expressions (2.5), (2.7) and (2.8), Equation (1.17) takes the form

$$\frac{\omega^2}{m^2}\frac{d}{d\psi}\left[\langle r^2 J\rangle \rho \frac{dX}{d\psi}\right] + \left\{-\omega^2 \rho \left\langle \frac{J}{r^2 H^2}\right\rangle + \gamma p \left\langle \frac{1}{J}\left(\frac{\partial J}{\partial \psi}\right)^2\right\rangle + \frac{dp}{d\psi}\frac{d\langle J\rangle}{d\psi} + \gamma p \left\langle \frac{\partial \ln J}{\partial \psi}\frac{\partial S}{\partial \chi}\right\rangle\right\} X = 0 \tag{2.9}$$

Here

$$\langle A\rangle = \frac{1}{2l}\oint \frac{A\, ds}{JH} = \frac{1}{2l}\int_{-l}^{l} A\, d\chi \tag{2.10}$$

Functions H and J in Equations (2.8) and (2.9) must satisfy the condition of equilibrium (2.4). When $8\pi p \ll H^2$, H and J differ only slightly from the corresponding distribution in the absence of a conducting medium. Since this difference is already taken into account in Equations (2.8) and (2.9), it is necessary to assume that H and J are the same as for the forceless configuration

$$\frac{\partial}{\partial\psi}(JH^2) = 0 \tag{2.11}$$

We will determine the boundary conditions for $X(\psi)$, taking into account that the plasma string occupies the volume $0 \leqslant \psi \leqslant \psi_0$. When $\psi = 0$ the function $X(\psi)$ must be bounded. If at $\psi = \psi_0$ the plasma string attaches to the wall, then

$$(X)_{\psi=\psi_0} = 0 \tag{2.12}$$

For a plasma string in vacuum the boundary condition , which follows from the continuity of total pressure on the perturbed surface of the plasma string, can be obtained directly from Equation (2.9). Let the density ρ and the pressure p change sharply from some finite values to 0 in a thin boundary layer $\psi_0 - \delta \leqslant \psi \leqslant \psi_0$ The thickness of this layer is considered to be small $(\delta \ll \psi_0)$, nevertheless we assume that the condition of applicability for the magnetohydrodynamic approximation is not disturbed. In this layer $dX/d\psi$ changes strongly while $X(\psi)$ is nearly constant. Functions J and H and their derivatives with respect to ψ in view of (2.9) will be assumed quantities in the layer. Integrating (2.10) we obtain

(for $\psi_0 - \delta \leqslant \psi \leqslant \psi_0$)

$$\left\{\frac{\omega^2\rho}{m^2}\langle r^2 J\rangle \frac{dX}{d\psi} + p\frac{d\langle J\rangle}{d\psi}X\right\}_{\psi=\psi_0-\delta} = 0 \tag{2.13}$$

Here the constant of integration is taken equal to zero since outside the layer $p = 0$ and $\rho = 0$. The problem has been reduced to finding those ω for which solutions (2.8) and (2.9) exist such, that (2.12) or (2.13) and the condition of finite X at zero are satisfied. For unstable oscillations $\omega^2 < 0$.

3. **Stability of a thin plasma string.** We will examine the equilibrium configuration for which the characteristic diameter of the plasma string $2r_0$ is much smaller than the length of one element of periodicity $2l$. The longitudinal magnetic field will be almost homogeneous and the function $\psi(r)$ will be approximately proportional to r^2. We will look for a solution of (2.3) and (2.11) in series form with respect to increments of the minor parameter $r_0{}^2/l^2$. Writing formally

$$r = \sqrt{\psi}\,[b(\chi) + \psi b_1(\chi) + \ldots], \qquad z = \chi - \psi b_2(\chi) + \ldots$$

we obtain from (2.3) and (2.11)

$$r = b\sqrt{\psi}\,[1 - {}^1/_8\psi(b'^2 + bb'')], \qquad z = \chi - {}^1/_2\psi bb' \tag{3.1}$$

$$H = 2b^{-2}(1 + {}^1/_2\psi bb''), \qquad J = {}^1/_2 b^2 \ (1 - \psi bb'')$$
$$r^2 H = 2\psi\,[1 - {}^1/_4\psi\,(b'^2 - bb'')] \tag{3.2}$$

Here $\psi b'^2 \sim r^2 / l^2$ is the minor parameter. Terms of the order $\psi^2 b'^4$ are discarded. Primes designate differentiation with respect to χ.

We will substitute expansions (3.1) and (3.2) into Equations (2.9) and (2.10). It is evident from (2.9) that the function S is of the order $b^3 b'$. Therefore the term with $\partial S/\partial\chi$ can be omitted in Equation (2.10). As a result we obtain

$$\psi \frac{d}{d\psi}\left(\psi\rho\,\frac{dX}{d\psi}\right) - m^2\rho\left\{\frac{1}{4} - \frac{3\psi\,\langle b^2 b'^2\rangle}{\omega^2\rho\langle b^4\rangle}\frac{dp}{d\psi}\right\}X = 0 \tag{3.3}$$

Corresponding to (2.13), the boundary condition in the case of a plasma string in a vacuum will be

$$\left\{\rho\psi\,\frac{dX}{d\psi} + \frac{3m^2\langle b^2 b'^2\rangle}{\omega^2\langle b^4\rangle}\,pX\right\}_{\psi=\psi_0-\delta} = 0 \tag{3.4}$$

In case of homogeneous distribution of temperature over the cross section when $p(\psi)$ is proportional to $\rho(\psi)$, a solution exists for (3.3) and (3.4)

$$X = \mathrm{const}\cdot\psi^{1/2 m}, \qquad \omega^2 = -\frac{6m\langle b^2 b'^2\rangle\,p}{\langle b^4\rangle\rho},\ m \geqslant 0 \tag{3.5}$$

The equation for ω^2 can be presented using average values for r, H and the radius of curvature R of a magnetic force line

$$\omega^2 = \frac{2mp}{\rho}\left(\int_{-l}^{l}\frac{d\chi}{H^2}\right)^{-1}\int_{-l}^{l}\frac{d\chi}{rRH^2} \qquad \left(R \approx \frac{1}{b''\sqrt{\psi}}\right) \tag{3.6}$$

In the case of an extended thin plasma string the integral which contains R is negative and Formula (3.6) corresponds to an unstable solution. This result is in qualitative agreement with conclusions in [1 and 4] in which fluted instabilities were investigated in analogy with instabilities of other systems.

In the case where the density is constant over the cross section of the plasma string (p = const, ρ = const for $\psi \leqslant \psi_0$ and $p = 0$, $\rho = 0$ for $\psi \geqslant \psi_0$) solution $X(t)$ in (3.5) will be a complete and bounded at zero solution of (3.3), and besides, ω^2 for the single unstable solution will be determined by Formula (3.5) (possible instabilities which arise in a detailed examination of the structure of the boundary layer of the plasma string are not taken into account). For other types of distributions of $p(\psi)$ other unstable oscillations can be distinguished. Some such solutions will be presented below.

4. Slowly developing instabilities. The substitution

$$S = -\frac{\gamma p}{\rho J H^2}\frac{\partial T}{\partial\chi} \tag{4.1}$$

brings the system (2.9) and (2.10) to the form

$$\frac{\gamma p}{\rho}\frac{\partial}{\partial\chi}\left(\frac{1}{JH^2}\frac{\partial T}{\partial\chi}\right)+\omega^2 JT-\frac{\omega^2 J}{\langle J\rangle}\langle JT\rangle=-J\frac{d\ln\langle J\rangle}{d\psi}+\frac{\partial J}{\partial\psi} \qquad (4.2)$$

$$\frac{d}{d\psi}\left\{\langle r^2J\rangle\rho\frac{dX}{d\psi}\right\}+m^2\left\{\frac{\rho\langle r^2J\rangle}{\omega^2}f(\psi)+\gamma p\left\langle\frac{\partial J}{\partial\psi}\left(T-\frac{\langle JT\rangle}{\langle J\rangle}\right)\right\rangle-\right.$$
$$\left.-\rho\left\langle\frac{J}{r^2H^2}\right\rangle\right\}X=0 \qquad (4.3)$$

$$f(\psi)=\frac{1}{\rho\langle r^2J\rangle}\frac{d\langle J\rangle}{d\psi}\left[\frac{dp}{d\psi}+\gamma p\frac{d\ln\langle J\rangle}{d\psi}\right] \qquad (4.4)$$

We will look for solution of these Equations which corresponds to instabilities with small increments (for $\omega^2\to 0$). In Equation (4.3) ω^2 will be minor parameter for the major derivative. Therefore, for this Equation strongly oscillating solutions are possible which can be satisfied by any boundary condition. For simplicity we assume that the surface $\psi = \psi_0$ is a wall and that boundary condition (2.12) is applicable.

Assuming that $X = U / \sqrt{\rho\langle r^2J\rangle}$, we will reduce (4.3) to such a form that the terms which characterize both singularities (for small ω and small ψ) are separated

$$\frac{d^2U}{d\psi^2}+\left\{\frac{m^2}{\omega^2}f(\psi)-\frac{m^2-1}{4\psi^2}+g(\psi)\right\}U=0 \qquad (4.5)$$

Here $g(\psi)$ is function which is finite for $\omega = 0$. Utilizing results from the previous section it is possible to show that for small ψ functions f and g are proportional to $1/\psi$, so that (4.5) is related to the type of equations for which the theory of asymptotic solutions (for $\omega\to 0$) is worked out [7]. If $f(\psi)$ is not close to zero then solution (4.5) which is finite at zero will be

$$U=\left(\frac{d\eta}{d\psi}\right)^{-1/2}V\left[\frac{m^2}{\omega^2}\eta(\psi)\right] \qquad (4.6)$$

$$\eta(\psi)=\left(\frac{1}{2}\int_0^{\psi}\sqrt{f(\psi)}\,d\psi\right)^2,\qquad V(x)=\Gamma(m+1)\sqrt{x}J_m(2\sqrt{x}) \qquad (m>0)$$

In the region of ψ close to ψ_0, the function $V(x)$ is proportional to $(2\sqrt{x}-2^{-1}m\pi-4^{-1}\pi)$, from condition $(U)_{\psi=\psi_0}=0$ we therefore obtain $2\sqrt{m^2\omega^{-2}\eta_0}\approx\pi q$, $\eta_0=\eta(\psi_0)$, where q is a large integer. For a different boundary condition q will not be an integer so that there is always solution*

$$\omega\approx\frac{2m\sqrt{\eta_0}}{\pi q},\qquad q\gg|m| \qquad (m\neq 0) \qquad (4.7)$$

The condition for stability ($\omega^2 > 0$) will be $\eta_0\geqslant 0$ or

$$f(\psi_0)\geqslant 0 \qquad (4.8)$$

* To determine the minimum value of parameter q it is necessary to perform a numerical integration of (4.3) for given boundary conditions.

which coincides with the known criterion of stability obtained in [1 to 3] if one takes into account that

$$\langle J \rangle = \int_{-l}^{l} H^{-1}\, ds$$

In the presence of zeros of the function $f(\psi)$ in the interval $0 < \psi < \psi_0$ an investigation of stability can also be easily carried out. Let for example $f(\psi) < 0$ for $0 < \psi < \psi_1$ and $f(\psi) < 0$ for $\psi_1 < \psi < \psi_0$, at the same time $f(\psi_1) = 0$, $(df/d\psi)_{\psi=\psi_1} \neq 0$. In the region of ψ not close to zero,

$$f(\psi) = (\psi - \psi_1)\, f^*(\psi - \psi_1), \quad f^*(0) \neq 0,$$

and here the asymptotic solution can be presented in the form [7]

$$U = \left(\frac{d\zeta}{d\psi}\right)^{-1/2} \{C_+ W_+ [\omega^{-2/3}\zeta(\psi)] + C_- W_- [\omega^{-2/3}\zeta(\psi)]\}$$

$$\zeta(\psi) = \left[\frac{3m}{2}\int_{\psi_1}^{\psi} \sqrt{f(\psi)}\, d\psi\right]^{2/3}, \qquad W_\pm(x) = \sqrt{x}\, J_{\pm 1/3}\left(\frac{2}{3}x^{3/2}\right) \tag{4.9}$$

The constants $C_\pm$ must be determined from the condition of continuity with solution (4.6) in the region $0 < \psi > \psi_1$, where both solutions are correct simultaneously and where asymptotic formulas can be used for the functions V and $W_\pm$. Taking into account that in the region $\psi_1 < \psi < \psi_0$ it is necessary to use different asymptotic formulas for $W_\pm(x)$ (because of change in sign of independent variable x in this region) substitution of the obtained result into boundary condition (2.12) leads again to Formula (4.7), in which $\eta_1 = \eta(\psi_1)$ just replaces $\eta_0 = \eta(\psi_0)$. Thus, fluted instabilities arise in the case when the function $f(\psi)$ is negative at least for some interval of values of ψ. These results are in agreement with conclusions in [1 to 3]. A formula analogous to [4.7] for the case of fluted instabilities in a plasma which is located in crossed magnetic and gravitational fields, was obtained in [8].

5. Stabilization of fluted instabilities for sufficiently large magnitude of Larmor radius of ions. We will examine the stability of a thin plasma string for the case where Larmor radius of ions a_j is so large that

$$\omega \sim \frac{a_j^2}{r_0^2}\omega_j, \qquad \omega_j = \frac{eH}{Mc} \tag{5.1}$$

Here M and e are respectively the mass and the charge of the ion, and ω_j is the cyclotron frequency of ions $(\omega_j \gg |\omega|)$. For condition (5.1) stabilization of fluted instabilities is possible [4]. Magnetohydrodynamic Equations have the form [9 and 6]

$$\rho\frac{d\mathbf{v}}{dt} = -\nabla p - \frac{1}{4\pi}\,\mathbf{H}\times \mathrm{rot}\,\mathbf{H} - \mathrm{div}\,(\boldsymbol{\pi} + \boldsymbol{\pi}^S) \tag{5.2}$$

$$\frac{\partial \mathbf{H}}{\partial t} = \mathrm{rot}\left\{\mathbf{v}\times\mathbf{H} - \frac{H}{\omega_j\rho}\nabla p_j\right\} \qquad \left(\begin{matrix}\rho = n_j M\\ p_j = n_j T_j\end{matrix}\right) \tag{5.3}$$

$$\frac{\partial\rho}{\partial t} + \mathrm{div}\,\rho\mathbf{v} = 0 \tag{5.4}$$

$$n_j\frac{dT_j}{dt} + (\gamma - 1)\,p_j\,\mathrm{div}\,\mathbf{v} + \mathrm{div}\left\{\frac{\gamma p_j}{\omega_j M H}(\mathbf{H}\times\nabla T_j)\right\} = 0 \tag{5.5}$$

where $n_j = n_e$ is the concentration of charged particles, T_j is the ionic temperature, $\boldsymbol{\pi} + \boldsymbol{\pi}^S$ is the tensor of viscous forces related to the tensor $\mathbf{W}\,(W_{ik} = \partial v_i/\partial y_k + \partial v_k/\partial y_i - (^2/_3)\,\delta_{ik}\,\mathrm{div}\, v$, y_i are Cartesian coordinates, the y_3-axis is along the magnetic field) [9] by Equations

$$\pi_{11} = -\pi_{22} = -\rho\nu W_{12}, \qquad \pi_{12} = \pi_{21} = {}^1/_2\, \rho\nu\,(W_{11} - W_{22})$$

$$\pi_{13} = \pi_{31} = -2\rho\nu W_{23}, \qquad \pi_{23} = \pi_{32} = 2\,\rho\nu W_{13} \tag{5.6}$$

$$\pi_{11}{}^S = \pi_{22}{}^S = -\rho\nu\omega_j\tau_j\,(W_{11} + W_{22}), \qquad \pi_{33}{}^S = -2\rho\nu\omega_j\tau_j W_{33}$$

$$\nu = \frac{1}{4}\,a_j^2\omega_j = \frac{T_j}{2\omega_j M} \tag{5.7}$$

Here δ_{ik} is the unit tensor; τ_j is the time of ion scattering on ions. Components of tensors π and π^S which are not written out are equal to zero. The magnetic field is assumed to be so strong that the parameter $\omega_j\tau_j \gg 1$. Formula (5.7) determines the viscosity of the plasma which depends on collisions between particles while the tensor (5.6) characterizes the effect of the finite magnitude of the Larmor radius of ions. The system of equations (5.2) to (5.5) must still be supplemented by an equation of state of the electron gas.

In the derivation of (5.3) Ohm's law is used in the form [9]

$$\mathbf{E} = -\frac{1}{c}\left\{\mathbf{v}\times\mathbf{H} - \frac{H}{\omega_j\rho}\left[\nabla p_j + \rho\,\frac{d\mathbf{v}}{dt} + \operatorname{div}(\pi + \pi^S)\right]\right\} \tag{5.8}$$

The term with div π leads to insignificant corrections [5], the inertial term for the condition $\omega_j \gg |\omega|$ is also small. In Equation (5.3) only the pressure term is taken into account $(p_j \sim \rho a_j^2\omega_j^2,\ \ p_j^* \sim p_j\xi\,/\,r_0,$ which is comparable to $r_0\rho\omega_j v^*)$.

We note that in arbitrary curvilinear orthogonal coordinates x_1, x_2 and x_3 following Formulas are appropriate [10]

$$W_{ik} = \frac{1}{h_k}\frac{\partial v_i}{\partial x_k} + \frac{1}{h_i}\frac{\partial v_k}{\partial x_i} - \frac{1}{h_i h_k}\left(v_i\,\frac{\partial h_i}{\partial x_k} + v_k\,\frac{\partial h_k}{\partial x_i}\right) + 2\delta_{ik}\left\{\sum_{q=1}^{3}\frac{v_q}{h_q}\frac{\partial \ln h_k}{\partial x_q} - \frac{\operatorname{div}\mathbf{v}}{3}\right\}$$

$$(\operatorname{div}\pi)_i = \frac{1}{h_i}\sum_{k=1}^{3}\left\{\frac{1}{h_1h_2h_3}\,\frac{\partial}{\partial x_k}\left(\frac{h_1h_2h_3h_i}{h_k}\,\pi_{ik}\right) - \frac{\partial \ln h_k}{\partial x_i}\,\pi_{kk}\right\} \tag{5.9}$$

Here h_i is the Lame coefficient, v_i are the orthogonal projections of velocity.

We will examine the stability of the equilibrium state described in section 2 with the additional assumptions that the temperature of ions and electrons is homogeneous over the cross section of the plasma string and that the length of an element of periodicity $2l$ is much greater than the characteristic diameter of the plasma string $2r_0$.

For condition $8\pi p \ll H^2$ we obtain (1.10) as a first approximation from (5.2). This equation is satisfied if the magnetic field remains unperturbed. Exactly such perturbations are the most dangerous ones and can lead to fluted instabilities. From linearized Equation (5.3) for $H^* = 0$ we obtain

$$X = X(\psi),\ Y = -\frac{dX}{d\psi} + \frac{2m\nu HT_j^*}{\omega T_j}\frac{d\ln n_j}{d\psi} \tag{5.10}$$

Here X, Y and Z are determined by Formulas (2.5). Writing Equation (5.5) for perturbed values and taking into account (5.10) we find

$$-\frac{T_j^*}{(\gamma-1)T_j}\left(1 - \frac{2m\gamma\nu H}{\omega n_j J}\frac{\partial n_j J}{\partial\psi}\right) = \operatorname{div}\xi$$
$$\operatorname{div}\xi = \frac{\partial\ln J}{\partial\psi}X + \frac{1}{J}\frac{\partial Z}{\partial\chi} + \frac{2m\nu HT_j^*}{\omega T_j}\frac{d\ln n_j}{d\psi} \tag{5.11}$$

If the order of the function $Z(\psi,\chi)$ is the same as for $\nu = 0$, then the following will hold

$$\frac{\partial\ln J}{\partial\psi}X \sim \frac{1}{J}\frac{\partial Z}{\partial\chi} \sim b'^2X \sim \frac{r_0^2X}{l^2\psi_0}$$

i.e. the perturbation of ion temperature is small and div ξ is close to zero. With an accuracy to corrections of the order r_0^2/l^2 we obtain

$$Y = -\frac{dX}{d\psi}, \qquad p_j^* = p_j\frac{n_j^*}{n_j}$$

Consequently, for perturbations which do not distort the magnetic field and for the condition $r_0^2 \ll l^2$, we obtain again Equations(2.8) which also which also turn out to be applicable even for $\nu \neq 0$. Substantially new effects can only be expected because the additional term $\operatorname{div}(\pi + \pi^S)$ enters into the equations of motion.

Due to the fact that the ionic temperature remains unperturbed in the first approximation and the energy exchange between electrons takes place faster than between ions, the perturbation of temperature $T_j + T_e$ will be insignificant and p^* will be proportional to ρ^*. This means that Equation (1.4) is applicable for $\gamma = 1$, otherwise the dependence on γ is not evident in the examined approximation (see Equation (3.3) section 3).

Thus the basic Equations take the form (1.16) and (1.17) to which terms containing $\operatorname{div}(\pi + \pi^S)^*$ must be added and into which Equation (2.8) must be substituted. From Formulas (5.9) we find

$$\frac{1}{2}(W_{11}^* - W_{22}^*) = i\omega\left(2\frac{dX}{d\psi} - \frac{X}{\psi}\right)$$

$$W_{12}^* = -\frac{2\omega\psi}{m}\left(\frac{d^2X}{d\psi^2} + \frac{m^2X}{4\psi^2}\right), \qquad W_{13}^* \sim W_{23}^* \sim \frac{\omega HZ}{r_0} \sim \frac{\omega X}{r_0 lH}$$

$$W_{11}^* + W_{22}^* \sim W_{33}^* \sim \frac{\omega X}{l^2H}, \qquad (\operatorname{div}\pi)_\chi^* \sim \frac{\omega\nu\rho X}{lr_0^2H}$$

$$(\operatorname{div}\pi)_\psi^* = \frac{\omega\nu}{mr}\left\{\frac{d}{d\psi}\left[\rho\left(4\psi^2\frac{d^2X}{d\psi^2} + m^2X\right)\right] - m^2\rho\left(2\frac{dX}{d\psi} - \frac{X}{\psi}\right)\right\}$$

$$(\operatorname{div}\pi)_\phi^* = \frac{2i\omega\nu}{r}\left\{\frac{d}{d\psi}\left[\rho\left(2\psi\frac{dX}{d\psi} - X\right)\right] - \psi\rho\left(\frac{d^2X}{d\psi^2} + \frac{m^2X}{4\psi^2}\right)\right\}$$

Here for an order of magnitude estimate of the function $Z(\psi,\chi)$ the equation was utilized which is obtained from the z-th component of Equation of motion (5.2); for this purpose it was assumed that the parameter $\omega\tau_j$ is of the order of one. For condition $|\omega\tau_j r_0^2/l^2| \ll 1$ components $(\operatorname{div}\pi^S)_\psi{}^*$ and $(\operatorname{div}\pi^S)_\varphi{}^*$ are small and instead of (1.17) the following holds:

$$\oint \frac{1}{H}\left\{\operatorname{div}\left[\frac{i\chi}{H}(\omega^2\rho\xi - \operatorname{div}\pi^*)\right] + \nabla(\gamma p \operatorname{div}\xi + \xi\cdot\nabla p)\cdot\operatorname{rot}\frac{i\chi}{H}\right\}ds = 0$$

From this we obtain the equation for $X(\psi)$ (terms containing $Z(\psi,\chi)$ do not enter into this equation for a first approximation)

$$\frac{\omega^2}{m^2}\left[\frac{d}{d\psi}\left(\psi\rho\frac{dX}{d\psi}\right) - \frac{m^2\rho'}{4\psi}X\right] + Q\frac{dp}{d\psi}X - \tag{5.12}$$

$$-\frac{2\omega\nu H}{m}\left\{\frac{d}{d\psi}\left(\psi\frac{d\rho}{d\psi}\frac{dX}{d\psi}\right) - \frac{1}{2}\frac{d^2\rho}{d\psi^2}X - \frac{m^2}{4\psi}\frac{d\rho}{d\psi}X\right\} = 0$$

$$Q = \frac{3\langle b^2b'^2\rangle}{\langle b^4\rangle} = -\left(\int_{-l}^{l}\frac{d\chi}{H^2}\right)^{-1}\int_{-l}^{l}\frac{d\chi}{rRH^2}$$

Equation (5.12) can be written in the form

$$\left(\omega^2 - 2m\omega\nu H\frac{d\ln\rho}{d\psi}\right)\left\{\psi\frac{d}{d\psi}\left(\psi\rho\frac{dX}{d\psi}\right) - \frac{m^2\rho}{4}X\right\} -$$

$$-2m\omega\nu H\psi^2\rho\frac{d^2\ln\rho}{d\psi^2}\frac{dX}{d\psi} + m^2\psi\left(Q\frac{dp}{d\psi} + \frac{\omega\nu H}{m}\frac{d^2\rho}{d\psi^2}\right)X = 0 \tag{5.13}$$

$$(\nu H = cT_j/2e = \text{const})$$

For a plasma string in vacuum the boundary condition can be obtained by integrating (5.12), however it is necessary to take now into account that in the layer $\psi_0 - \delta \leqslant \psi \leqslant \psi_0$ only the derivative $d\rho/d\psi$ can change sharply but not $\rho(\psi)$. The region in which $\rho(\psi)$ changes sharply must be related to the internal region. The boundary condition is written in the form

$$\left(\psi\frac{dX}{d\psi} - \frac{X}{2}\right)_{\psi=\psi_0-\delta} = 0 \tag{5.14}$$

For perturbations with $m = 1$ solution (3.5) satisfies Equation (5.12) and condition (5.14) for any arbitrary $\rho(\psi)$. Consequently, these perturbations remain unstable also for large values of Larmor radii of ions. This conclusion was obtained in [4] in which the specific case of density distribution was examined

$$\rho = \frac{\rho^\circ}{p^\circ}p = \rho^\circ e^{-\lambda\psi}, \qquad p^\circ,\ \rho^\circ,\ \lambda = \text{const} \tag{5.15}$$

Let us examine the problem of stability for the distribution (5.15). Solution (5.13) is presented by means of degenerate hypergeometric function

$$X = \text{const}\cdot\psi^{1/2m} F(A, m+1, \lambda\psi) \tag{5.16}$$

$$A = \frac{m}{2} + \frac{m^2\rho^{-1}pQ - m\omega\nu H\lambda}{\omega^2 + 2m\omega\nu H\lambda} \qquad (m>0)$$

It will be required that for large values of $\lambda\psi$ the function $X(\psi)$ be bounded. This condition leads, as was shown in [11], approximately to the same equality which is the prerequisite for non-exponential growth of perturbations for large $\lambda\psi$. We obtain $A \approx -n$, for $n = 0, 1, 2, \ldots$ or

$$\frac{\omega}{m} = -\nu H\lambda\left(1 - \frac{1}{m+2n}\right) \pm \left\{\left[\nu H\lambda\left(1 - \frac{1}{m+2n}\right)\right]^2 - \frac{2pQ}{(m+2n)\rho}\right\}^{1/2} \tag{5.17}$$

The stability condition for the most dangerous perturbations with $n = 0$ will be

$$\left[\nu H\lambda\left(1 - \frac{1}{m}\right)\right]^2 \geqslant \frac{2pQ}{m\rho} \tag{5.18}$$

Formulas (5.17) and (5.18) are in qualitative agreement with approximate calculations in [4]. The parameter λ in these calculations can be represented by the quantity ψ_0 which characterizes the diameter of the plasma string ($\lambda\psi_0 \sim 1$). The quantities $\nu H = cT_j / 2e = \text{const}$ and $p / \rho = \text{const}$.

In conclusion, we will examine the stability with respect to slowly increasing instabilities which were investigated in section 4. Reducing Equation (5.13) to the form (4.5) we see that instead of the term f/ω^2 which characterizes the behavior of solutions for $\omega \to 0$, there is now $D(\psi)$ where

$$D(\psi) = \frac{Q}{\psi\rho\,(\omega^2 - 2m\omega\nu H d\ln\rho / d\psi)}\frac{dp}{d\psi}$$

It is assumed here that $\omega\nu H/\psi$, in accordance with condition (5.1) is of the order of ω^2 and ω^2 is so small that $\omega^2 l^2 \ll p / \rho$.

Instead of (4.7) we obtain dispersion Equation

$$m\int_0^{\psi_0} \sqrt{D(\psi)}\, d\psi \approx \pi q, \qquad q \gg |m| \qquad (m \neq 0) \tag{5.19}$$

For an order of magnitude estimate we take the expression under the integral sign outside of the integral for some intermediate value of ψ Then we obtain

$$\frac{\omega}{m} = \nu H\frac{d\ln\rho}{d\psi} \pm \left[\left(\nu H\frac{d\ln\rho}{d\psi}\right)^2 + \frac{\psi_0^2 Q}{\pi^2 q^2 \psi\rho}\frac{dp}{d\psi}\right]^{1/2} \tag{5.20}$$

Here $dp/d\psi$ is negative, therefore for $\nu = 0$ Formula (5.20) describes an unstable solution.

The criterion for stability will be

$$\left(\nu H\frac{d\ln\rho}{d\psi}\right)^2 \geqslant -\frac{\psi_0^2 Q}{\pi^2 q^2 \psi\rho}\frac{dp}{d\psi}, \qquad q \gg |m| \qquad (m \neq 0) \tag{5.21}$$

Since, as far as the order of magnitude is concerned, the increment of slowly increasing instabilities is q times $(q \gg 1)$ smaller than the increment of instability given by Formula (3.6), the stability criterion (5.21) is less severe than (5.18). For the examined perturbations stabilization of instabilities with $m = 1$ is also possible.

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